

Quantum trajectories

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Abstract

This paper presents a new approach to phase space trajectories in quantum mechanics. A Moyal description of quantum theory is used, where observables and states are treated as classical functions on a classical phase space. A quantum trajectory being an appropriate solution to quantum Hamiltonian equations is also a function defined on a classical phase space. It results in a deformation of a classical action of a flow on observables and states to an appropriate quantum action. It also leads to a new multiplication rule for any quantum trajectory treated as a one-parameter group of diffeomorphisms. Moreover, several examples are given, presenting the developed formalism for particular quantum systems.

Keywords: quantum mechanics, deformation quantization, quantum trajectories, canonical transformations, Moyal product

1. Introduction

The time evolution of a classical Hamiltonian system is fully given by trajectories (a flow) in a phase space. Having calculated a classical flow Φ_t for the given system a time evolution of states and observables can be received by simply composing them with Φ_t . A classical flow is defined as a map $\Phi_t: M \rightarrow M$ on the phase space M , which at every point $\xi_0 \in M$ gives a trajectory (curve) $\gamma(t) = \Phi_t(\xi_0)$ on M passing through the point ξ_0 and being a solution of the Hamilton's equations. Geometrically trajectories constitute a flow of a Hamiltonian vector field. Furthermore, any trajectory $\Phi_t(\xi_0)$ has the property of being a canonical transformation for every t , and the set $\{\Phi_t\}_{t \in \mathbb{R}}$ have a structure of a group with multiplication being a composition of maps.

From the very beginning of quantum physics, efforts have been taken to formulate some kind of an analogue of phase space trajectories in quantum mechanics [1]. The most common approaches to quantum dynamics are the de Broglie-Bohm approach [2–4], the average value approach [5, 6], and the Moyal trajectories approach (see [7, 8] and references therein). Worth noting is also the paper [9] written by Rieffel where he considers a classical limit of a quantum time evolution in the framework of a strict deformation quantization. In the following paper we develop the Moyal approach to time evolution. The usual formulation of the theory of Moyal trajectories is based on the phase space description of quantum mechanics, where one considers the Heisenberg evolution of fundamental observables of position and momentum, being \hbar -deformation of the classical Hamiltonian evolution. Moreover, the deformation of arbitrary order can be calculated by an \hbar -hierarchy of recursive first order linear partial differential equations [7, 8, 10]. The time evolution of observables cannot be given as a simple composition of observables with a quantum flow. For this reason Dias and Prata [7], and Krivoruchenko and Faessler [8] considered observables as \star -functions and a quantum phase space as a plane of noncommuting variables. Then the action of a flow on observables was given as a \star -composition.

In our approach to quantum trajectories we treat observables as ordinary functions on a classical phase space. We present in explicit form a quantum action of a flow on observables, which is a deformation of

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the respective classical action. The resulting time dependence of observables gives an appropriate solution of a quantum time evolution equation for observables (1) (Heisenberg's representation on a phase space). Then, we show that a set of quantum symplectomorphisms (quantum flow) has a structure of a group with multiplication (quantum composition) being a deformation of the ordinary composition considered as a multiplication in a group of classical symplectomorphisms (classical flow). The explicit form of the quantum composition law is presented. Such approach to quantum trajectories have a benefit in that it is not needed to calculate the form of observables as \star -functions, but only a quantum action of a given trajectory needs to be found. Also we expect that our approach to quantum flows will allow a development of phase space quantum mechanics in a geometrical setting similar to that of classical Hamiltonian mechanics.

The paper is organized as follows. In Section 2 we review a basics of a quantum mechanics on a phase space. Section 3 contains the main results of a theory of quantum trajectories. Finally, in Section 4, examples of particular quantum systems are presented.

2. Phase space quantum mechanics

2.1. Preliminaries

The most natural approach to quantum theory, when dealing with quantum trajectories, is a phase space quantum mechanics (see [11], and [12–14] for recent reviews). The following review of phase space quantum mechanics comes from [15]. The phase space approach to quantum theory is based on an appropriate deformation of a classical Hamiltonian mechanics, with respect to some parameter which we take to be the Planck's constant \hbar . The deformation of a classical Hamiltonian system can be fully given by deforming an algebraic structure of a classical Poisson algebra. This will then yield a deformation of a phase space (a Poisson manifold) to a noncommutative phase space (a noncommutative Poisson manifold), a deformation of classical states to quantum states and a deformation of classical observables to quantum observables.

First, let us deal with a deformation of a phase space. A Poisson manifold (M, \mathcal{P}) (\mathcal{P} being a Poisson tensor) is fully described by a Poisson algebra $\mathcal{A}_C = (C^\infty(M), \cdot, \{\cdot, \cdot\})$ of smooth complex-valued functions on the phase space M , where \cdot is a point-wise product of functions and $\{\cdot, \cdot\}$ is a Poisson bracket induced by a Poisson tensor \mathcal{P} . Hence by deforming \mathcal{A}_C to some noncommutative algebra $\mathcal{A}_Q = (C^\infty(M), \star, [\cdot, \cdot])$, where \star is some noncommutative associative product of functions being a deformation of a point-wise product, and $[\cdot, \cdot]$ is a Lie bracket satisfying the Leibniz's rule and being a deformation of the Poisson bracket $\{\cdot, \cdot\}$, we can think of a quantum Poisson algebra \mathcal{A}_Q as describing a noncommutative Poisson manifold.

The algebra \mathcal{A}_C contains in particular a subset of classical observables, whereas \mathcal{A}_Q contains a subset of quantum observables. Note that quantum observables are functions on the phase space M similarly as in classical mechanics. Furthermore, classical observables are real-valued functions from \mathcal{A}_C , i.e., self-adjoint functions with respect to the complex-conjugation — an involution in the algebra \mathcal{A}_C . Quantum observables should also be self-adjoint functions with respect to an involution in the algebra \mathcal{A}_Q . However, in general the complex-conjugation do not need to be an involution in \mathcal{A}_Q . Thus in \mathcal{A}_Q we have to introduce some involution which would be a deformation of the complex-conjugation [15]. As a consequence, quantum observables (self-adjoint functions with respect to the quantum involution) might be complex and \hbar -dependent.

There is a vast number of equivalent quantization schemes (see [15] for review of the subject) which yield a quantization equivalent to a standard approach to quantum mechanics but giving different orderings of position and momentum operators. From this diversity of quantization schemes the simplest one is a Moyal quantization. It follows from the fact that for the Moyal quantum algebra the involution is the complex-conjugation as in the classical case. Thus in this case quantum observables, exactly like classical observables, can be chosen as real-valued functions. Further on we will deal only with that distinguished quantization. Such a choice is not a restriction as other quantization schemes known in the literature are gauge equivalent to the Moyal one (see [15] and Subsection 2.3).

The Moyal quantization scheme is as follows. First, let us assume that $M = \mathbb{R}^{2N}$ and $\mathcal{P} = \partial_{x^i} \wedge \partial_{p_i}$. Define a \star -product by a formula

$$f \star g = f \exp \left(\frac{1}{2} i \hbar \overleftarrow{\partial}_{x^i} \overrightarrow{\partial}_{p_i} - \frac{1}{2} i \hbar \overleftarrow{\partial}_{p_i} \overrightarrow{\partial}_{x^i} \right) g.$$

This \star -product is called the Moyal product. For a two-dimensional case ($N = 1$) the Moyal product reads

$$f \star g = \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{i\hbar}{2} \right)^k \sum_{m=0}^k \binom{k}{m} (-1)^m (\partial_x^{k-m} \partial_p^m f) (\partial_x^m \partial_p^{k-m} g).$$

The deformed Poisson bracket $\llbracket \cdot, \cdot \rrbracket$ associated with the \star -product will be given in terms of a \star -commutator $[\cdot, \cdot]$ as follows

$$\llbracket f, g \rrbracket = \frac{1}{i\hbar} [f, g] = \frac{1}{i\hbar} (f \star g - g \star f), \quad f, g \in \mathcal{A}_Q.$$

To avoid problems with convergence of the series in the above definition of the \star -product the common practice is to extend the space $C^\infty(M)$ to a space $C^\infty(M)[\hbar]$ of formal power series in \hbar with coefficients from $C^\infty(M)$. The \star -product is then properly defined on such space.

Observe that every function $f \in \mathcal{A}_Q$ can be expanded into a \star -power series

$$f = \sum_{n,m=0}^{\infty} a_{nm} \underbrace{x \star \cdots \star x}_n \star \underbrace{p \star \cdots \star p}_m,$$

where $a_{nm} \in \mathbb{C}$. Indeed, the result follows from the fact that every monomial $x^n p^m$ can be written as a \star -polynomial, which on the other hand can be seen from the definition of the \star -product.

2.2. Space of states, expectation values of observables and time evolution equation

In general a space of states is fully characterized by the algebraic structure of the quantum Poisson algebra \mathcal{A}_Q [16, 17]. It can be shown that for the Moyal quantization states can be represented as quantum distribution functions, i.e., square integrable functions ρ defined on the phase space satisfying certain conditions [15, 18]. For this reason the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^{2N})$ of square integrable functions on the phase space will be called a space of states. Observe, that the Moyal product can be extended to a product between smooth functions from $C^\infty(\mathbb{R}^{2N})$ and square integrable functions from $L^2(\mathbb{R}^{2N})$.

Formulas for the expectation values of observables and the time evolution of states are similar as in classical mechanics, except that the point-wise product \cdot of functions and the Poisson bracket $\{\cdot, \cdot\}$ have to be replaced with the \star -product and the deformed Poisson bracket $\llbracket \cdot, \cdot \rrbracket$. Thus the expectation value of an observable $A \in \mathcal{A}_Q$ in a state $\rho \in L^2(\mathbb{R}^{2N})$ is given by the formula

$$\begin{aligned} \langle A \rangle_\rho &= \iint (A(0) \star \rho(t))(x, p) \, dx \, dp = \langle A(0) \rangle_{\rho(t)} \\ &= \iint (A(t) \star \rho(0))(x, p) \, dx \, dp = \langle A(t) \rangle_{\rho(0)}. \end{aligned}$$

The time evolution equation of quantum distribution functions $\rho(t)$ (Schrödinger picture) is the counterpart of the Liouville's equation describing the time evolution of classical distribution functions, and is given by the formula

$$\frac{d\rho}{dt} - \llbracket H, \rho(t) \rrbracket = 0 \quad \Leftrightarrow \quad i\hbar \frac{d\rho}{dt} - [H, \rho(t)] = 0,$$

where H is a Hamiltonian (distinguished observable from \mathcal{A}_Q). The time evolution of quantum observable $A(t)$ (Heisenberg picture) is given by

$$\frac{dA}{dt} - \llbracket A(t), H \rrbracket = 0 \quad \Leftrightarrow \quad i\hbar \frac{dA}{dt} - [A(t), H] = 0. \quad (1)$$

2.3. Equivalence of quantizations

Two star-products \star and \star' are said to be gauge equivalent if there exists a vector space automorphism $S: C^\infty(\mathbb{R}^{2N}) \rightarrow C^\infty(\mathbb{R}^{2N})$ of the form

$$S = \sum_{k=0}^{\infty} \hbar^k S_k, \quad S_0 = 1, \quad (2)$$

where S_k are linear operators, which satisfies the formula

$$S(f \star g) = S f \star' S g, \quad f, g \in C^\infty(\mathbb{R}^{2N}).$$

If, moreover, the automorphism S preserves the deformed Poisson brackets and involutions $*$ and $*'$ from the algebras $\mathcal{A}_Q = (C^\infty(\mathbb{R}^{2N}), \star, \llbracket \cdot, \cdot \rrbracket, *)$ and $\mathcal{A}'_Q = (C^\infty(\mathbb{R}^{2N}), \star', \llbracket \cdot, \cdot \rrbracket', *)$, i.e.,

$$S(\llbracket f, g \rrbracket) = \llbracket S f, S g \rrbracket', \quad S(f^*) = (S f)^{*'},$$

then S is an isomorphism of the algebra \mathcal{A}_Q onto the algebra \mathcal{A}'_Q .

Two quantizations of a classical Hamiltonian system are equivalent if there exists an isomorphism S of their quantum Poisson algebras. This equivalence is mathematical as well as physical. It has been stressed out in Subsection 2.1, that to the same measurable quantity correspond different functions from respective quantum Poisson algebras. This observation seems to be missing in considerations of different quantizations present in the literature. In fact, to every observable $A \in \mathcal{A}_Q$ from one quantization scheme corresponds an observable $A' = S A \in \mathcal{A}'_Q$ from the other quantization scheme. Both observables A and A' describe the same measurable quantity and in the limit $\hbar \rightarrow 0$ reduce to the same classical observable. Such approach to equivalence of quantum systems introduces, indeed, physically equivalent quantizations as the functions A , A' from different quantization schemes have the same spectra, expectation values, etc., and when they are Hamiltonians they describe the same time evolution.

It is possible to define a morphism of spaces of states of different quantization schemes, in terms of S . This morphism we will also denote by S . In case when the initial quantization is the Moyal quantization S will be a Hilbert space isomorphism. In what follows we will restrict to the case when the S -image of the space of states $L^2(\mathbb{R}^{2N})$ is also a Hilbert space $L^2(\mathbb{R}^{2N}, \mu)$ of square integrable functions possibly with respect to a different measure μ .

3. Quantum trajectories in phase space

As before we will consider the Moyal quantization of a classical Hamiltonian system (M, \mathcal{P}, H) , where $M = \mathbb{R}^{2N}$, $\mathcal{P} = \partial_{x^i} \wedge \partial_{p_i}$, and $H \in C^\infty(M)$ is an arbitrary real function.

The solution of quantum Hamiltonian equations

$$\dot{Q}^i(t) = \llbracket Q^i(t), H \rrbracket, \quad \dot{P}_j(t) = \llbracket P_j(t), H \rrbracket, \quad (3)$$

where $Q^i(x, p, 0) = x^i$ and $P_j(x, p, 0) = p_j$, i.e., the Heisenberg representation (1) for observables of position and momentum, generates a quantum flow Φ_t in phase space according to an equation

$$\Phi_t(x, p; \hbar) = (Q(x, p, t; \hbar), P(x, p, t; \hbar)). \quad (4)$$

For every instance of time t the map Φ_t is a quantum canonical transformation (quantum symplectomorphism) from coordinates x, p to new coordinates $x' = Q(x, p, t; \hbar)$, $p' = P(x, p, t; \hbar)$. In other words Φ_t preserves the quantum Poisson bracket: $\llbracket Q^i(t), P_j(t) \rrbracket = \delta_j^i$ (this can be easily seen from (7) and the fact that $\llbracket Q^i(0), P_j(0) \rrbracket = \llbracket x^i, p_j \rrbracket = \delta_j^i$).

The flow Φ_t , as every other quantum canonical transformation, can act on observables and states as simple composition of maps. Such classical action can also be used to transform the algebraic structure

of the quantum Poisson algebra so that the action will be an isomorphism of the initial algebra and its transformation. A star-product \star_t being the Moyal product transformed by Φ_t is defined by the formula

$$(f \star g) \circ \Phi_t^{-1} = (f \circ \Phi_t^{-1}) \star_t (g \circ \Phi_t^{-1}), \quad f, g \in C^\infty(\mathbb{R}^{2N}). \quad (5)$$

The \star_t -product takes the form of the Moyal product but with derivatives ∂_{x^i} , ∂_{p_i} replaced by some other derivations D_{x^i} , D_{p_i} of the algebra $C^\infty(\mathbb{R}^{2N})$:

$$f \star_t g = f \exp \left(\frac{1}{2} i \hbar \overleftarrow{D_{x^i}} \overrightarrow{D_{p_i}} - \frac{1}{2} i \hbar \overleftarrow{D_{p_i}} \overrightarrow{D_{x^i}} \right) g,$$

where derivations D_{x^i} , D_{p_i} are transformations of the derivatives ∂_{x^i} , ∂_{p_i} :

$$(\partial_{x^i} f) \circ \Phi_t^{-1} = D_{x^i}(f \circ \Phi_t^{-1}), \quad (\partial_{p_i} f) \circ \Phi_t^{-1} = D_{p_i}(f \circ \Phi_t^{-1}).$$

The \star_t -product can be also written in a different form, a so called covariant form. For more details see e.g. [11, 19, 20]. The crucial point of our construction is the observation that for a wide class of quantum flows the \star_t -product is gauge equivalent to the Moyal product. Strictly speaking, to a quantum flow Φ_t there corresponds a unique isomorphism S_t of the form (2) satisfying

$$S_t(f \star g) = S_t f \star_t S_t g, \quad (6a)$$

$$S_t x^i = x^i, \quad S_t p_j = p_j, \quad (6b)$$

$$S_t(f^*) = (S_t f)^*. \quad (6c)$$

We will consider only such flows to which an isomorphism S_t can be associated, however, we believe that this holds for every quantum flow. Note, that for the \star_t -algebra the involution is also the complex-conjugation.

A formal solution of the time evolution equation (1) for an observable $A \in \mathcal{A}_Q$ can be expressed by the formula

$$A(t) = e^{-t[[H, \cdot]]} A(0) = e_{\star}^{\frac{i}{\hbar} t H} \star A(0) \star e_{\star}^{-\frac{i}{\hbar} t H},$$

where

$$e^{-t[[H, \cdot]]} := \sum_{k=0}^{\infty} \frac{1}{k!} (-t)^k \underbrace{[[H, [H, \dots [H, \cdot] \dots]]]}_k$$

and

$$e_{\star}^{\frac{i}{\hbar} t H} := \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{i}{\hbar} t \right)^k \underbrace{H \star \dots \star H}_k.$$

In particular, the solution of (3) takes the form

$$Q^i(t) = e^{-t[[H, \cdot]]} Q^i(0) = e_{\star}^{\frac{i}{\hbar} t H} \star Q^i(0) \star e_{\star}^{-\frac{i}{\hbar} t H}, \quad (7a)$$

$$P_j(t) = e^{-t[[H, \cdot]]} P_j(0) = e_{\star}^{\frac{i}{\hbar} t H} \star P_j(0) \star e_{\star}^{-\frac{i}{\hbar} t H}, \quad (7b)$$

which for fixed initial condition $Q^i(x, p, 0) = x^i$ and $P_j(x, p, 0) = p_j$ represents a particular quantum trajectory.

A time evolution of an observable $A \in \mathcal{A}_Q$ should be alternatively expressed by action of the quantum flow Φ_t on A . The composition of Φ_t with observables (the classical action of Φ_t on observables) does not result in a proper time evolution of observables. Thus it is necessary to deform this classical action. We will prove that a proper action of the quantum flow Φ_t on functions from \mathcal{A}_Q (a pull-back of Φ_t) is given by the formula

$$\Phi_t^* A = (S_t A) \circ \Phi_t, \quad (8)$$

where S_t is an isomorphism associated to the quantum canonical transformation Φ_t^{-1} . Indeed, (8) can be proved first by noting that

$$\Phi_t^* Q^i(0) = (S_t Q^i(0)) \circ \Phi_t = Q^i(0) \circ \Phi_t = Q^i(t) = e^{-t\llbracket H, \cdot \rrbracket} Q^i(0)$$

and similarly

$$\Phi_t^* P_j(0) = e^{-t\llbracket H, \cdot \rrbracket} P_j(0),$$

where the fact that $S_t x^i = x^i$ and $S_t p_j = p_j$ was used, which on the other hand was a consequence of the quantum canonicity of Φ_t . Secondly, Φ_t^* given by (8) is an automorphism of \mathcal{A}_Q as

$$\Phi_t^*(A \star B) = (S_t(A \star B)) \circ \Phi_t = (S_t A \star_t S_t B) \circ \Phi_t = ((S_t A) \circ \Phi_t) \star ((S_t B) \circ \Phi_t) = \Phi_t^* A \star \Phi_t^* B,$$

where \star_t denotes a star-product transformed by Φ_t^{-1} . Thus

$$\Phi_t^* = e^{-t\llbracket H, \cdot \rrbracket} \quad (9)$$

holds true since every function from \mathcal{A}_Q can be expressed as a \star -power series (see Subsection 2.1).

In a complete analogy with classical theory one can define a quantum Hamiltonian vector field by $\zeta_H = \llbracket \cdot, H \rrbracket$. Then (9) states that Φ_t is a flow of the quantum Hamiltonian vector field ζ_H . Also in an analogy with classical mechanics $\{\Phi_t\}$ is a one-parameter group of quantum canonical transformations with respect to a multiplication defined by

$$\Phi_{t_1} \Phi_{t_2} = (S_{t_2} \Phi_{t_1}) \circ \Phi_{t_2}, \quad (10)$$

where $S_{t_2} \Phi_{t_1}$ denotes a map $\mathbb{R}^{2N} \rightarrow \mathbb{R}^{2N}$ given by the formula

$$S_{t_2} \Phi_{t_1} = (S_{t_2} Q^1(t_1), \dots, S_{t_2} P_N(t_1)),$$

where $\Phi_{t_1} = (Q^1(t_1), \dots, Q^N(t_1), P_1(t_1), \dots, P_N(t_1))$. Multiplication defined in such a way satisfies properties similar to their classical counterparts:

$$\Phi_0 = \text{id}, \quad \Phi_{t_1} \Phi_{t_2} = \Phi_{t_1+t_2},$$

proving that $\{\Phi_t\}$ is a group. Further on we will call it a quantum composition. The quantum composition rule given by (10) is properly defined since it respects the quantum pull-back of flows:

$$(\Phi_{t_1} \Phi_{t_2})^* = \Phi_{t_2}^* \circ \Phi_{t_1}^*. \quad (11)$$

Indeed, it is enough to show (11) for an arbitrary \star -monomial. For simplicity we will present the proof for a two-dimensional case and for a \star -monomial $x \star p$. Using the fact that $S_t x = x$ and $S_t p = p$ for every t , following from quantum canonicity of the flow Φ_t , one calculates that

$$\begin{aligned} (\Phi_{t_2}^* \circ \Phi_{t_1}^*)(x \star p) &= \Phi_{t_2}^*((S_{t_1}(x \star p)) \circ \Phi_{t_1}) = \Phi_{t_2}^*((x \star_{t_1} p) \circ \Phi_{t_1}) = \Phi_{t_2}^*(Q(t_1) \star P(t_1)) \\ &= (S_{t_2}(Q(t_1) \star P(t_1))) \circ \Phi_{t_2} = (S_{t_2} Q(t_1) \star_{t_2} S_{t_2} P(t_1)) \circ \Phi_{t_2} \\ &= (x \star_{t_2, t_1} p) \circ S_{t_2} \Phi_{t_1} \circ \Phi_{t_2}, \end{aligned}$$

where \star_{t_1} , \star_{t_2} , denote Moyal products transformed, respectively, by transformations $\Phi_{t_1}^{-1}$, $\Phi_{t_2}^{-1}$, and \star_{t_2, t_1} denotes the \star_{t_2} -product transformed by $(S_{t_2} \Phi_{t_1})^{-1}$. Now, from the relation $S_{T_1 \circ T_2} = S_{T_1, T_2} S_{T_1}$ valid for any transformations T_1, T_2 defined on the whole phase space ($S_{T_1 \circ T_2}$ is an isomorphism intertwining star-products \star and $\star_{T_1 \circ T_2}$, S_{T_1, T_2} intertwines \star_{T_1} with $\star_{T_1 \circ T_2}$, and S_{T_1} intertwines \star with \star_{T_1} , where \star_{T_1} and $\star_{T_1 \circ T_2}$ are Moyal products transformed, respectively, by transformations T_1 and $T_1 \circ T_2$), one receives that

$$S_{(\Phi_{t_1} \Phi_{t_2})^{-1}}(x \star p) = S_{\Phi_{t_2}^{-1}, (S_{t_2} \Phi_{t_1})^{-1}} S_{t_2}(x \star p) = S_{\Phi_{t_2}^{-1}, (S_{t_2} \Phi_{t_1})^{-1}}(x \star_{t_2} p) = x \star_{t_2, t_1} p.$$

Hence

$$(\Phi_{t_2}^* \circ \Phi_{t_1}^*)(x \star p) = S_{(\Phi_{t_1}, \Phi_{t_2})^{-1}}(x \star p) \circ S_{t_2} \Phi_{t_1} \circ \Phi_{t_2} = (\Phi_{t_1} \Phi_{t_2})^*(x \star p).$$

In the limit $\hbar \rightarrow 0$, (7) reduces to classical phase space trajectories

$$\begin{aligned} Q^i(t) &= e^{-t\{H, \cdot\}} Q^i(0), & P_j(t) &= e^{-t\{H, \cdot\}} P_j(0), \\ Q^i(x, p, 0) &= x^i, & P_j(x, p, 0) &= p_j, \end{aligned}$$

which are formal solutions of classical Hamiltonian equations

$$\dot{Q}^i(t) = \{Q^i(t), H\}, \quad \dot{P}_j(t) = \{P_j(t), H\}. \quad (12)$$

In more explicit form classical trajectories are represented by a flow (diffeomorphism)

$$\Phi_t(x, p) = (Q(x, p, t), P(x, p, t)), \quad (13)$$

which is an $\hbar \rightarrow 0$ limit of the quantum flow (4). Diffeomorphism (13) is a classical symplectomorphism. An action of the classical flow Φ_t on functions from \mathcal{A}_C (a pull-back of Φ_t) is just a simple composition of functions with Φ_t , being an $\hbar \rightarrow 0$ limit of (8)

$$\Phi_t^* A = A \circ \Phi_t. \quad (14)$$

$\{\Phi_t\}$ forms a one-parameter group of canonical transformations, preserving a classical Poisson bracket: $\{Q^i(t), P_j(t)\} = \delta_j^i$, with a multiplication being an ordinary composition of maps

$$\Phi_{t_1} \Phi_{t_2} = \Phi_{t_1} \circ \Phi_{t_2}, \quad (15)$$

which is the $\hbar \rightarrow 0$ limit of (10).

4. Examples

4.1. Example 1: Harmonic oscillator

In this example we will consider quantum trajectories of the harmonic oscillator. The Hamiltonian of the harmonic oscillator is given by the equation

$$H(x, p) = \frac{1}{2} (p^2 + \omega^2 x^2).$$

It happens that in such case the quantum trajectory coincides with the classical one. Indeed, one can show that

$$\begin{aligned} Q(t) &= e^{-t\llbracket H, \cdot \rrbracket} Q(0) = e^{-t\{H, \cdot\}} Q(0), \\ P(t) &= e^{-t\llbracket H, \cdot \rrbracket} P(0) = e^{-t\{H, \cdot\}} P(0) \end{aligned}$$

and in explicit form classical/quantum trajectory $\Phi_t = (Q(t), P(t))$ of a harmonic oscillator is

$$\begin{aligned} Q(x, p, t) &= x \cos \omega t + \omega^{-1} p \sin \omega t, \\ P(x, p, t) &= p \cos \omega t - \omega x \sin \omega t. \end{aligned}$$

Observe that the classical action (composition) of Φ_t on the algebra of observables preserves the Moyal product, i.e.,

$$(f \star g) \circ \Phi_t = (f \circ \Phi_t) \star (g \circ \Phi_t), \quad f, g \in C^\infty(\mathbb{R}^{2N}).$$

Thus in accordance with (6) the unique isomorphism S_t associated with Φ_t is equal $S_t = 1$. This means that the action of the flow Φ_t on observables (8) as well as the quantum composition rule (10) for the flow is equal to the classical composition rule of that flow. In other words the time evolution of observables is the same as in classical case. The difference between the classical and quantum system is in the admissible states which evolve along the flow. In classical case states are probabilistic distribution functions, whereas in quantum case states are quasi-probabilistic distribution functions. In particular, classical pure states are Dirac distribution functions; however, quantum pure states will no longer be of such form due to the Heisenberg uncertainty principle.

4.2. Example 2

In this example let us consider a two particle system described by the Hamiltonian

$$H(x, p) = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} + kx^1 p_2^2,$$

where m_1 and m_2 are masses of particles and k is a coupling constant. The solution of quantum Hamiltonian equations (3) reads [20]

$$\begin{aligned} Q^1(t) &= x^1 + \frac{1}{m_1} p_1 t - \frac{k}{2m_1} p_2^2 t^2, \\ P_1(t) &= p_1 - k p_2^2 t, \\ Q^2(t) &= x^2 + \left(\frac{1}{m_2} p_2 + 2kx^1 p_2 \right) t + \frac{k}{m_1} p_1 p_2 t^2 - \frac{k^2}{3m_1} p_2^3 t^3, \\ P_2(t) &= p_2, \end{aligned}$$

which coincides again with a solution of classical Hamiltonian equations. However, in accordance with (5) the received quantum flow Φ_t transforms the Moyal product to the following product

$$f \star_t g = f \exp \left(\frac{1}{2} i \hbar \overleftarrow{D}_{x^i} \overrightarrow{D}_{p_i} - \frac{1}{2} i \hbar \overleftarrow{D}_{p_i} \overrightarrow{D}_{x^i} \right) g,$$

where

$$\begin{aligned} D_{x^1} &= \partial_{x^1} + 2ktp_2 \partial_{x^2}, \\ D_{p_1} &= \partial_{p_1} + \frac{1}{m_1} t \partial_{x^1} + \frac{k}{m_1} t^2 p_2 \partial_{x^2}, \\ D_{x^2} &= \partial_{x^2}, \\ D_{p_2} &= \partial_{p_2} - 2ktp_2 \partial_{p_1} - \frac{k}{m_1} t^2 p_2 \partial_{x^1} + \left(\frac{1}{m_2} t + 2ktx^1 - \frac{k}{m_1} t^2 p_1 - \frac{k^2}{m_1} t^3 p_2^2 \right) \partial_{x^2}. \end{aligned}$$

Moreover, the isomorphism S_t associated with Φ_t and intertwining the Moyal product with the \star_t -product takes the form

$$S_t = \exp \left(\frac{1}{8} \hbar^2 \frac{k}{m_1} t^2 \partial_{x^1} \partial_{x^2}^2 + \frac{1}{4} \hbar^2 kt \partial_{p_1} \partial_{x^2}^2 + \frac{1}{12} \hbar^2 \frac{k^2}{m_1} t^3 p_2 \partial_{x^2}^3 \right).$$

Indeed, a direct calculations show that the relations (6) are satisfied. More details of the construction of S_t the reader can find in [21].

As in this case $S_{t_2} \Phi_{t_1} = \Phi_{t_1}$, the group multiplication for $\{\Phi_t\}$ is just a composition of maps, as one could expect since Φ_t is simultaneously the classical and quantum trajectory. However, the action of Φ_t on observables and states does not reduce in general to a composition of maps (14). This shows that the time evolution of quantum observables differs in general from the time evolution of classical observables.

One can check by direct calculations that the action of the quantum flow Φ_t on an observable A , given by (8), indeed describes the quantum time evolution of A . As an example let us take $A(x, p) = x_1 x_2^2$. Then

$$(S_t A)(x, p) = x_1 x_2^2 + \frac{1}{4} \hbar^2 \frac{k}{m_1} t^2$$

and it can be checked that

$$A(t) = (S_t A) \circ \Phi_t = Q^1(t) (Q^2(t))^2 + \frac{1}{4} \hbar^2 \frac{k}{m_1} t^2$$

satisfies the time evolution equation (1).

4.3. Example 3

In this example we will consider a system described by a Hamiltonian

$$H(x, p) = x^2 p^2.$$

The solution of quantum Hamiltonian equations (3) reads [7]

$$Q(x, p, t; \hbar) = \sec^2(\hbar t) x \exp\left(\frac{2}{\hbar} \tan(\hbar t) xp\right), \quad (16a)$$

$$P(x, p, t; \hbar) = \sec^2(\hbar t) p \exp\left(-\frac{2}{\hbar} \tan(\hbar t) xp\right), \quad (16b)$$

for $|t| < \frac{\pi}{2\hbar}$. This solution is a deformation of a classical one given by the limit $\hbar \rightarrow 0$

$$Q_C(x, p, t) = x e^{2txp}, \quad P_C(x, p, t) = p e^{-2txp}.$$

The induced quantum flow Φ_t is an example of a flow for which Φ_t , for every $t \in (-\frac{\pi}{2\hbar}, \frac{\pi}{2\hbar}) \setminus \{0\}$, is not a classical symplectomorphism, since

$$\{Q(t), P(t)\} = \sec^4(\hbar t) \neq 1.$$

In accordance with (5) the quantum flow Φ_t transforms the Moyal product to the following product

$$f \star_t g = f \exp\left(\frac{1}{2} i \hbar \overleftarrow{D}_x \overrightarrow{D}_p - \frac{1}{2} i \hbar \overleftarrow{D}_p \overrightarrow{D}_x\right) g,$$

where

$$\begin{aligned} D_x &= \sec^2(\hbar t) (1 + 2ta(\hbar t)xp) \exp(2ta(\hbar t)xp) \partial_x - 2t \sec^2(\hbar t) a(\hbar t) p^2 \exp(2ta(\hbar t)xp) \partial_p, \\ D_p &= 2t \sec^2(\hbar t) a(\hbar t) x^2 \exp(-2ta(\hbar t)xp) \partial_x + \sec^2(\hbar t) (1 - 2ta(\hbar t)xp) \exp(-2ta(\hbar t)xp) \partial_p, \end{aligned}$$

and $a(x) = \frac{\tan(x)}{x \sec^4(x)}$. Moreover, the isomorphism S_t associated with Φ_t and intertwining the Moyal product with the \star_t -product, up to the second order in \hbar , takes the form

$$\begin{aligned} S_t &= 1 + \hbar^2 \left(\frac{1}{6} (3t^2 x^3 + 4t^3 x^4 p) \partial_x^3 + \frac{1}{6} (3t^2 p^3 - 4t^3 x p^4) \partial_p^3 + \frac{1}{2} (-tp - t^2 x p^2 + 4t^3 x^2 p^3) \partial_x \partial_p^2 \right. \\ &\quad \left. + \frac{1}{2} (tx - t^2 x^2 p - 4t^3 x^3 p^2) \partial_x^2 \partial_p + (2t^2 x^2 + 2t^3 x^3 p) \partial_x^2 + (2t^2 p^2 - 2t^3 x p^3) \partial_p^2 + (-2t^2 xp) \partial_x \partial_p \right) + o(\hbar^4). \end{aligned} \quad (17)$$

Indeed, expanding relations (6) with respect to \hbar one can prove that S_t in the above form satisfies these relations up to $o(\hbar^2)$.

From the fact that Φ_t is a purely quantum trajectory, we deal with the quantum group multiplication (10) for $\{\Phi_t\}$ as well as the quantum action (8) of Φ_t on observables and states. Indeed, expanding (16) with respect to \hbar :

$$\begin{aligned} Q(x, p, t; \hbar) &= Q_C \left(1 + \hbar^2 \left(t^2 + \frac{2}{3} t^3 xp \right) \right) + o(\hbar^4), \\ P(x, p, t; \hbar) &= P_C \left(1 + \hbar^2 \left(t^2 - \frac{2}{3} t^3 xp \right) \right) + o(\hbar^4) \end{aligned}$$

and applying isomorphism S_t (17), the quantum composition law

$$\begin{aligned} Q(t_1 + t_2) &= S_{t_2} Q(t_1) \circ \Phi_{t_2} = S_{t_1} Q(t_2) \circ \Phi_{t_1}, \\ P(t_1 + t_2) &= S_{t_2} P(t_1) \circ \Phi_{t_2} = S_{t_1} P(t_2) \circ \Phi_{t_1} \end{aligned}$$

holds up to $o(\hbar^2)$. Note also, that the flow Φ_t is not defined for all $t \in \mathbb{R}$ but only on an interval $(-\frac{\pi}{2\hbar}, \frac{\pi}{2\hbar})$, contrary to classical flows which are always globally defined. This is an interesting result showing that in general the quantum time evolution do not have to be defined for all instances of time t .

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